AP Calculus BC
Review: Sequences, Infinite Series, and Convergence

Sequences

A **sequence** $\{a_n\}$ is a function whose domain is the set of positive integers.

- The functional values $a_1, a_2, a_3, \ldots a_n$ are called the **terms** of the sequence. The number $\boldsymbol{a_n}$ is called the **nth term** of the sequence.

- A sequence is either **convergent** (if the sequence has a limit, i.e. it approaches a specific number) or **divergent** (if the sequence does not approach a specific limit)

- Finding the limit of an infinite sequence utilizes the following theorem

  i) If $\{a_n\}$ is a sequence and f is a function such that f(n) = a_n for all positive integers n, **AND**

  ii) if f(x) is defined for all real numbers x ≥ 1, **AND**

  iii) if $\lim_{x\to\infty} f(x)$ exists, then $\lim_{n\to\infty} a_n = \lim_{x\to\infty} f(x)$

The theorem above allows us to evaluate limits of sequences at infinity by using the results for evaluating limits of functions at infinity. Especially useful will be L'Hopital's Rule for indeterminate forms (0/0 or ∞/∞)

Example 1 : Find the limit of the sequence $\left(\frac{3n}{e^{5n}}\right\}$.

First note that the first three terms are $\frac{3}{e^5}, \frac{6}{e^{10}}, \frac{9}{e^{15}}$. This sequence is obviously approaching 0 as n approaches infinity. Let's apply the theorem anyway.

Let f(x) = $\frac{3x}{e^{5x}}$, then

$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{3x}{e^{5x}} = \frac{\infty}{\infty}$ so L'Hopital's Rule can be applied.

$\lim_{x\to\infty} \frac{3x}{e^{5x}} = \lim_{x\to\infty} \frac{3}{5e^{5x}} = 0$, therefore (as we predicted) the sequences converges to 0.

Example 2: Show that the sequence $\left\{\frac{1+2n^3}{n^3}\right\}$ converges.

Again, let's look at the first couple of terms.

$3, \frac{17}{8}, \frac{55}{27}$

Looks like the terms are converging to 2.

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty}\left(\frac{1+2n^3}{n^3}\right) = \lim_{x\to\infty}\left(\frac{1+2x^3}{x^3}\cdot\frac{\frac{1}{x^3}}{\frac{1}{x^3}}\right) = \lim_{x\to\infty}\left(\frac{\frac{1}{x^3}+2}{1}\right) = \frac{2}{1} = 2$$

We will often have to find a formula for the general, or nth, term of a sequence. Look at the next example.

Example 3: If the first four terms of a sequence $\{a_n\}$ are $1, \frac{9}{7}, \frac{27}{11}, \frac{81}{15}$

a) find a formula for the nth term of the sequence

b) determine whether the sequence converges or diverges

First, let's rewrite the sequnece as $\frac{3^1}{3}, \frac{3^2}{7}, \frac{3^3}{11}, \frac{3^4}{15}$

Now, notice that the denominators are one less than multiples of 4. So the sequence can be written

$\frac{3^1}{4(1)-1}, \frac{3^2}{4(2)-1}, \frac{3^3}{4(3)-1}, \frac{3^4}{4(4)-1},$

therefore, the nth term is $a_n = \frac{3^n}{4n-1}$

b) Let $f(x) = \frac{3^x}{4x-1}$

since $\lim_{x\to\infty} f(x) = \frac{\infty}{\infty}$ we can apply L'Hopital's Rule

$\lim_{x\to\infty}\frac{3^x}{4x-1} = \lim_{x\to\infty}\frac{3^x(\ln 3)}{4} = \infty$. Therefore, the sequence diverges.

Series

If $\{a_n\}$ is a sequence, then $S_n = a_1 + a_2 + a_3 + \ldots + a_n$ is called the nth partial sum, $\{S_n\}$ is a sequence of partial sums. For example,

$S_1 = a_1,\; S_2 + a_1 + a_2,\; S_3 = a_1 + a_2 + a_3, S_n = a_1 + a_2 + a_3 + \ldots + a_n.$

For the infinite series which is denoted $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \ldots + a_n + \ldots$

- If $\lim_{n\to\infty} S_n$exists and is equal to the number L, then the series $\sum_{n=1}^{\infty} a_n$ converges and L is the sum of the series.

- If $\lim_{n\to\infty} S_n$ is nonexistent, then the series $\sum_{n=1}^{\infty} a_n$ diverges and has no sum.

We will often be asked to determine whether an infinite series converges or diverges and, if is converges, what is the series' sum?

It is often difficult, if not impossible, to find the sum of an infinite series. So, let's concentrate on just determining whether or not a series converges. We will list the most frequently used tests to determine convergence. They are listed below.

1. nth term test

For the infinite series $\sum_{n=1}^{\infty} a_n$

i) if $\lim_{n\to\infty} a_n \neq 0$, then the series diverges.

ii) CAUTION! If $\lim_{n\to\infty} a_n = 0$, the series does not necessarily converge.

iii) if the series does converge, then $\lim_{n\to\infty} a_n = 0$.

Do not be confused with the nth term test. What is says is that if the limit of the nth term is not 0, then the series diverges. However, the fact that the limit of the nth term is 0 does not mean that it necessarily converges.

Example 4: Determine whether $\sum_{n=1}^{\infty} 2^n$ converges or diverges.

Let's look at the first three terms,

2, 4, 8

Obviously, each successive term is greater than the previous so $\lim_{n\to\infty} 2^n = \infty$. Since $\lim_{n\to\infty} a_n \neq 0$, the series diverges.

2. Specific Series Type

A) **The geometric series $\sum_{n=1}^{\infty} a \cdot r^{n-1}$** (a ≠ 0)

i) converges if | r | < 1 and has a sum $\frac{a}{1-r}$

ii) diverges if | r | ≥ 1

Example 5: Determine whether $\sum_{n=1}^{\infty} \frac{5}{3^n}$ converges or diverges.

$$\sum_{n=1}^{\infty} \frac{5}{3^n} = \sum_{n=1}^{\infty} 5 \cdot \frac{1}{3^n} = \sum_{n=1}^{\infty} 5 \cdot \frac{1}{3 \cdot 3^{n-1}} = \sum_{n=1}^{\infty} \frac{5}{3} \left(\frac{1}{3}\right)^{n-1}.$$

This is a geometric series with a = $\frac{5}{3}$ and r = $\frac{1}{3}$.

Since | r | = | $\frac{1}{3}$ | < 1, the series converges. In this case, we can easily find the sum of this series.

$$\text{sum} = \frac{a}{1-r} = \frac{\frac{5}{3}}{1-\frac{1}{3}} = \frac{5}{2}$$

B) **The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$**

i) converges if p > 0

ii) diverges if p ≤ 1

Example 6: Determine the convergence or divergence of $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \ldots + \frac{1}{n^2} + \ldots$

$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \ldots + \frac{1}{n^2} + \ldots = \sum_{n=1}^{\infty} \frac{1}{n^2}$.

This is a p-series with p = 2. Since p > 1, the series converges.

Example 7: Determine the convergence or divergence of $\sum_{n=1}^{\infty} \frac{1}{\sqrt[7]{n^4}}$

$\sum_{n=1}^{\infty} \frac{1}{\sqrt[7]{n^4}} = \sum_{n=1}^{\infty} \frac{1}{n^{4/7}}$.

This is a p-series with p = $\frac{4}{7}$, Since p < 1, the series diverges.

C) **The harmonic series** $\boldsymbol{\sum_{n=1}^{\infty} \frac{1}{n}}$ diverges (notice this is really just a p-series with p = 1)

Example 8: Determine the convergence or divergence of $\sum_{n=1}^{\infty} \frac{3}{n}$.

$\sum_{n=1}^{\infty} \frac{3}{n} = 3 \cdot \sum_{n=1}^{\infty} \frac{1}{n}$

Since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, so does $3 \cdot \sum_{n=1}^{\infty} \frac{1}{n}$

D) **The alternating series** $\boldsymbol{\sum_{n=1}^{\infty} (-1)^{n-1} a_n}$ **or** $\boldsymbol{\sum_{n=1}^{\infty} (-1)\, a_n}$ converges if :

i) $0 < a_{n+1} < a_n$, i.e. the series is non-increasing **AND**

ii) $\lim_{n\to\infty} a_n = 0$ (note that is condition by itself is not sufficient to determine convergence but, when used in conjunction with a non-increasing alternating series, the two conditions determine convergence)

Example 9: Determine the convergence or divergence of $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{3n}$.

$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{3n} = \sum_{n=1}^{\infty} (-1)^{n+1} \cdot \frac{1}{3n}$ with $a_n = \frac{1}{3n}$.

We must verify two conditions:

1) $a_{n+1} < a_n$ and 2) $\lim_{n\to\infty} a_n = 0$

$\frac{1}{3(n+1)} < \frac{1}{3n}$?? $\lim_{n\to\infty} \frac{1}{3n} = 0$ True ✓

3n < 3(n + 1)

3n < 3n + 3

0 < 3 True ✓

Therefore, the series converges by the alternating series test

3. Ratio Test

For the series $\sum_{n=1}^{\infty} a_n$ find $\left| \frac{a_{n+1}}{a_n} \right| = L$

i) if L < 1, then the series converges absolutely

ii) if L > 1 (or L is infinite), then the series diverges

iii) if L = 1, the test is inconclusive - must try another test

Hint: Try the ratio test if a_n contains factors such as n! or x^n.

Note: A series for which $\sum_{n=1}^{\infty} |a_n|$ converges is called an absolutely convergent series. In addition, if the series is absolutely convergent, then it is also just convergent. However, if a series $\sum_{n=1}^{\infty} a_n$ is convergent, but the series $\sum_{n=1}^{\infty} |a_n|$ is divergent, then the series $\sum_{n=1}^{\infty} a_n$ is called conditionally convergent.

An example of this is the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$, which is convergent by the alternating series test but the series $\sum_{n=1}^{\infty} \left| \frac{(-1)^{n+1}}{n} \right| = \sum_{n=1}^{\infty} \frac{1}{n}$ is a divergent harmonic series. So the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ is a conditionally convergent series.

Example 10: Determine the convergence or divergence of $\sum_{n=1}^{\infty} \frac{5^n}{n!}$.

With $a_n = \frac{5^n}{n!}$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{\frac{5^{n+1}}{(n+1)!}}{\frac{5^n}{n!}} \right| = \lim_{n\to\infty} \left| \frac{5^{n+1}}{(n+1)!} \cdot \frac{n!}{5^n} \right| = \lim_{n\to\infty} \left| \frac{5}{n+1} \right| = 0 = \text{L}$$

Since L < 1, the series converges absolutely.

Example 11: Determine the convergence or divergence of $\sum_{n=1}^{\infty} \frac{1}{n^4}$

With $a_n = \frac{1}{n^4}$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{1}{(n+1)^4} \cdot \frac{n^4}{1} \right| = \lim_{n\to\infty} \left| \frac{n^4}{(n+1)^4} \right| = 1 = \text{L}$$

With L = 1, the Ratio Test is inconclusive. However, the series $\sum_{n=1}^{\infty} \frac{1}{n^4}$ is just a p-series with p = 4 >1, so the given series converges absolutely.

4) Root Test

For the series $\sum_{n=1}^{\infty} a_n$ find $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \text{L}$

i) if L < 1 then the sereis converges absolutely

ii) if L > 1 (or L is infinite), then the series diverges

iii) if L = 1, the test is inconclusive and another test may be used

Note: The Root Test is used infrequently on the AP Exam

Example 12: Determine the convergence or divergence of $\sum_{n=1}^{\infty} \frac{e^{3n}}{n^n}$

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left|\frac{e^{3n}}{n^n}\right|} = \lim_{n\to\infty} \left(\frac{e^{3n}}{n^n}\right)^{\frac{1}{n}}$$

We can drop the absolute value since all terms are positive.

$$= \lim_{n\to\infty} \left(\frac{e^3}{n}\right) = 0 = L$$

Since L < 1, the given series converges absolutely.

Example 13: Determine the convergence or divergence of $\sum_{n=1}^{\infty} \frac{1}{(\ln n)^n}$.

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left|\frac{1}{(\ln n)^n}\right|} = \lim_{n\to\infty} \sqrt[n]{\frac{1}{(\ln n)^n}} = \lim_{n\to\infty} \left(\frac{1}{(\ln n)^n}\right)^{\frac{1}{n}}$$

$$= \lim_{n\to\infty} \frac{1}{\ln n} = 0 = L$$

Since L < 1, the given series converges absolutely.

Example 14: Determine the convergence or divergence of $\sum_{n=1}^{\infty} \frac{(n+1)^n}{(2n+1)^n}$.

$$\lim_{n\to\infty} \sqrt[n]{\left|\frac{(n+1)^n}{(2n+1)^n}\right|} = \lim_{n\to\infty} \sqrt[n]{\frac{(n+1)^n}{(2n+1)^n}} = \lim_{n\to\infty} \left(\frac{n+1}{2n+1}\right) = 2 = L$$

Since L > 1, the given series diverges by the Root Test

5. Integral Test

For the series $\sum_{n=1}^{\infty} a_n$, where $a_n = f(n)$ and f(n) is positive, continuous, and decreasing for x ≥ 1:

i) if the improper integral $\int_1^{\infty} f(x)\, dx$ exists, then the series $\sum_{n=1}^{\infty} a_n$ converges.

ii) if the improper integral $\int_1^{\infty} f(x)\, dx = \infty$, then the series $\sum_{n=1}^{\infty} a_n$ diverges.

Example 15: Determine the convergence or divergence of $\sum_{n=1}^{\infty} \frac{1}{3n+1}$.

Let f(x) = $\frac{1}{3x+1}$ (Note that f(x) is positive, continuous, and decreasing for x ≥ 1.

$\int_1^{\infty} \frac{1}{3x+1}\, dx = \lim_{c\to\infty} \int_1^{c} \frac{1}{3x+1}\, dx = \lim_{c\to\infty} \frac{1}{3} \int_1^{c} \frac{3}{3x+1}\, dx = \lim_{c->\infty} \frac{1}{3}[\ln|3x+1|]_1^{c}$

$= \frac{1}{3} \lim_{c\to\infty} [\ln|3x+1|] = \frac{1}{3} \lim_{c\to\infty} [\ln|3c+1| - \ln 4] = \infty$

Therefore, the given series diverges by the integral test.

6. Comparison Test

Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be a series of positive terms:

i) if $\sum_{n=1}^{\infty} b_n$ is a known convergent series and $a_n \le b_n$ for all positive n, then the series $\sum_{n=1}^{\infty} a_n$ is convergent.

ii) if $\sum_{n=1}^{\infty} b_n$ is a known divergent series and $a_n \ge b_n$ for all positive n, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.

Example 16: Determine the convergence or divergence of $\sum_{n=1}^{\infty} \frac{5}{3+2^n}$.

Let $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{5}{3+2^n}$; consider the series $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{5}{2^n}$

1. $a_n = \frac{5}{3+2^n} < \frac{5}{2^n} = b_n$

2. $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{5}{2^n} = \sum_{n=1}^{\infty} 5 \cdot \left(\frac{1}{2}\right)^n$ This is a convergent geometric series with r = $\frac{1}{2} < 1$. Since $a_n < b_n$ and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

7. Limit Comparison Test

Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be series of positive terms.

Find $\lim_{n\to\infty} \frac{a_n}{b_n} = L$.

i) if L > 0, then the series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ either both converge or both diverge.

ii) If L = 0, and if $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

iii) if L = ∞ and if $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

Example 17: Determine the convergence or divergence of $\sum_{n=1}^{\infty} \frac{1}{4n^2-5n+3}$.

Choose the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$, which is a convergent p-series (p = 2 > 1) with $a_n = \frac{1}{4n^2-5n+3}$ and $b_n = \frac{1}{n^2}$,

$$\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \lim_{n\to\infty}\left(\frac{1}{4n^2-5n+3}\cdot\frac{n^2}{1}\right) = \lim_{n\to\infty}\left(\frac{n^2}{4n^2-5n+3}\right) = \frac{1}{4} > 0.$$

So the given series converges by the limit comparison test.

Summary of Tests for Convergence or Divergence of Infinite Series

Test	Series	Necessary Conditions	Conclusion
nth term	$\sum_{n=1}^{\infty} a_n$	$\lim_{n\to\infty} a_n \neq 0$	diverges
Geoemtric Series	$\sum_{n=1}^{\infty} a_n$	$\mid r \mid < 0$	converges
		$\mid r \mid \geq 0$	diverges
p-Series	$\sum_{n=1}^{\infty} \frac{1}{p^n}$	p > 1	converges
		p ≤ 1	diverges
Harmonic Series	$\sum_{n=1}^{\infty} \frac{1}{n}$		diverge
Alternating Series	$\sum_{n=1}^{\infty} (-1)^{n-1} a_n$	$0 < a_{n+1} \leq a_n$ and $\lim_{n\to\infty} a_n = 0$	converges
Ratio	$\sum_{n=1}^{\infty} a_n$	$\lim_{n\to\infty} \left\lvert \frac{a_{n+1}}{a_n} \right\rvert < 1$	converges
		$\lim_{n\to\infty} \left\lvert \frac{a_{n+1}}{a_n} \right\rvert > 1$	diverges
		$\lim_{n\to\infty} \left\lvert \frac{a_{n+1}}{a_n} \right\rvert = 1$	inconclusive
Root	$\sum_{n=1}^{\infty} a_n$	$\lim_{n\to\infty} \sqrt[n]{\mid a_n \mid} < 1$	converges
		$\lim_{n\to\infty} \sqrt[n]{\mid a_n \mid} > 1$	diverges
		$\lim_{n\to\infty} \sqrt[n]{\mid a_n \mid} = 1$	inconclusive
Integral	$\sum_{n=1}^{\infty} a_n$	$a_n = f(n)$and f is continuous, positive, and increasing	
		$\int_1^{\infty} f(x)\, dx$ exists	$\sum_{n=1}^{\infty} a_n$ converges
		$\int_1^{\infty} f(x)\, dx$ does not exists	$\sum_{n=1}^{\infty} a_n$ diverges

Comparison	$\sum_{n=1}^{\infty} a_n$	$0 < a_n \leq b_n$ and $\sum_{n=1}^{\infty} b_n$ converges	$\sum_{n=1}^{\infty} a_n$ converges
		$a_n \geq b_n > 0$ and $\sum_{n=1}^{\infty} b_n$ diverges	$\sum_{n=1}^{\infty} a_n$ diverges
Limit Comparison	$\sum_{n=1}^{\infty} a_n$	$\lim_{n\to\infty} \frac{a_n}{b_n} = \mathrm{L} > 0$ and $\sum_{n=1}^{\infty} b_n$ conv.	$\sum_{n=1}^{\infty} a_n$ converges
		$\lim_{n\to\infty} \frac{a_n}{b_n} = \mathrm{L} > 0$ and $\sum_{n=1}^{\infty} b_n$ diver.	$\sum_{n=1}^{\infty} a_n$ diverges

Alternating Series Error Bound

If the sum of a convergent alternating series is approximated by using the sum of the first N terms of the series, an error is introduced. A theorem in calculus states that the absolute value of this error is less than the value of the N + 1st term. In other words, if the sum of the series is S and we approximate the series' sum by S_n(the sum of the first N terms), then the error R_n is given by:

$|\mathrm{S} - S_n| = |R_n| \leq a_{n+1}$

Example 18: Approximating the sum of the following series by its first 5 terms

$\sum_{n=1}^{\infty} (-1)^n \left(\frac{1}{n^2}\right) = -1 + \frac{1}{4} - \frac{1}{9} + \frac{1}{16} - \frac{1}{25} + \frac{1}{36} - \frac{1}{49} \ldots$

The sum of the first 5 terms is : $-1 + \frac{1}{4} - \frac{1}{9} + \frac{1}{16} - \frac{1}{25} \approx -.838611$

The given series is convergent by the Alternating Series Test since

$a_{n+1} = \frac{1}{(n+1)^2} \leq \frac{1}{n^2} = a_n$ and $\lim_{n\to\infty} a_n = 0$

The error bound is given by :$|\mathrm{S} - S_5| = |R_5| \leq \frac{1}{36} \approx .027778$

So the actual sum lies between S_5 - .027778 and S_5 + .027778; therefore the sum, S, of the series lies in the interval

$-.838611 - .027778 \leq \mathrm{S} \leq -.838611 + .027778$

$-.866389 \leq \mathrm{S} \leq -.810833$

Radius and Interval of Convergence of Power Series

The infinite series discussed so far, $\sum_{n=1}^{\infty} a_n$ have involved sums of constant terms where $a_n = f(n)$ for some function f. We now investigate infinite series whose terms are constant multiples of x - c, with c being a specific number.

A power series centered at c is a series of the form

$\sum_{n=1}^{\infty} a_n \cdot (x-c)^n = a_0 \cdot (x-c)^0 + a_1 \cdot (x-c)^1 + a_2 \cdot (x-c)^2 + a_3 \cdot (x-c)^3 + \ldots + a_n \cdot (x-c)^n + \ldots$

$= a_0 + a_1 \cdot (x-c) + a_2 \cdot (x-c)^2 + a_3 \cdot (x-c)^3 + \ldots + a_n \cdot (x-c)^n + \ldots$

For each value of the variable x, the power series represents a series of numbers whose convergence or divergence can be determined using the tests previously discussed

- For any power series $\sum_{n=1}^{\infty} a_n \cdot (x-c)^n$, exactly one of the following conditions is true:

i) the series is only convergent when x = c

ii) the series is absolutely convergent for all x

OR

iii) there exists some positive number R for which the series converges absolutely for x such that |x - c| < R and diverges for all x such that |x - c| > R

- The number R is called the radius of convergence of the power series:

i) if the series is only convergent at x = c, then the radius of convegence is R = 0.

ii) if the series is convergent for all x, the radius of convergence is infinity

- The set of all numbers x for which the series is convergent is called the interval of convergence of the power series. As we shall soon see, a power series may converge at both end points, at just one endpoint, or at both endpoints of the interval of convergence. The interval of convergence may be in one of the following forms:

[c - R, c + R] or (c - R, c + R) or (c - R, c + R] or [c - R, c + R)

Example 19: Find the radius of convergence for the power series $\sum_{n=0}^{\infty} x^n$

With $a_n = x^n$, using the Ratio Test, we have

$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{x^{n+1}}{x^n} \right| = \lim_{n\to\infty} | x | = |x|$

If the limit above is less than 1, then the series converges absolutely. We solve |x| < 1 to get -1 < x < 1.

Next, check for convergence at the endpoints

1. When x = 1, series is $\sum_{n=0}^{\infty} (1)^n = 1 + 1 + 1 + \ldots$which diverges

2. When x = -1, series is $\sum_{n=0}^{\infty} (-1)^n = 1 - 1 + 1 - 1 + \ldots$ which diverges

After excluding both endpoints, the interval of convergence is (-1, 1): the radius of convergence is 1.

Example 20: Find the interval of convergence for the series $\sum_{n=0}^{\infty} \frac{x^n}{n}$.

With $a_n = \frac{x^n}{n}$, using the Ratio Test again:

$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{x^{n+1}}{n+1} \cdot \frac{n}{x^n} \right| = \lim_{n\to\infty} \left| \frac{n}{n+1} \cdot x \right| = 1 \cdot |x| = |x|$

When this limit is less than 1, the series converges: |x| < 1 ⇒ -1 < x < 1. Check for convergence at the end-points:

1. When x = 1, the series is $\sum_{n=0}^{\infty} \frac{(1)^n}{n} = \sum_{n=0}^{\infty} \frac{1}{n}$, which is the divergent harmonic series.

2. When x = -1, the series is $\sum_{n=0}^{\infty} \frac{(-1)^n}{n}$, which converges by the alternatin series test.

So we exclude x = 1, but include x = -1. Therefore the interval of convergence is $-1 \le x < 1$ or [-1, 1)