AP Calculus BC
Review and Worksheet: Alternating Series

An **alternating series** is a series whose terms are alternately positive and negative. Here are two examples:

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdot \cdot \cdot = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$$

$$-\frac{1}{2} + \frac{2}{3} - \frac{3}{4} + \frac{4}{5} - \frac{5}{6} + \frac{6}{7} - \cdot \cdot \cdot = \sum_{n=1}^{\infty} (-1)^{n} \frac{n}{n+1}$$

We see from these examples that the *n*th term of an alternating series is of the form

$$a_n = (-1)^{n-1} b_n \qquad \text{or} \qquad a_n = (-1)^{n} b_n$$

where b_n is a positive number (in facy, $b_n = |a_n|$.)

The following test says that if the terms of an alternating sereis decreases toward zero in absolute value, then the series converges.

The Alternating Series Test If the alternating series

$$\sum_{n=1}^{\infty} (-1)^{n-1} b_n = b_1 - b_2 + b_3 - b_4 + b_5 - b_6 + \cdot \cdot \cdot \qquad b_n > 0$$

satisfies

(i) $b_{n+1} \le b_n$ for all n

(ii) $\lim_{n\to\infty} b_n = 0$

then the series is convergent.

Example 1: Show that the alternating harmonic series converges.

$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdot \cdot \cdot = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$

(i) $b_{n+1} < b_n$ because $\frac{1}{n+1} < \frac{1}{n}$

(ii) $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{n} = 0$

so the alternating harmonic series is convergent by the Alternating Series Test

Problem 1: Determine whether the series $\sum_{n=1}^{\infty} \frac{(-1)^n\, 3\, n}{4\, n - 1}$ converges.

Problem 2: Test the series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{n^3 + 1}$ for convergence or divergence.

Estimating Sums: A partial sum s_n of any convergent series can be used as an approximation to the total sums, but this is not of much use unless we can estimate the accuracy of the approximation. The error involved in using $s \approx s_n$ is the remainder $R_n = s - s_n$.The following theorem says that for series that satisfy the conditions of the Alternating Series Test, the size of the error is smaller than b_{n+1}, which is the absolute value of the first neglected term.

Alternating Series Estimation Theorem

If s = $\Sigma\,(-1)^{n-1}\,b_n$ is the sum of an alternating series that satisfies

(i) $0 \leq b_{n+1} \leq b_n$ and (ii) $\lim_{n\to\infty} b_n = 0$

then $|R_n| = |s - s_n| \leq b_{n+1}$

Caution: This rule is, in general, valid only for alternating series that satisfy the conditions of the Alternating Series Estimation Theorem. **The rule does not apply to other type of series.**

Example 2: Find the sum of the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n!}$ correct to three decimal places.

We first observe thst the series is convergent by the Alternating Series Test because

(i) $\frac{1}{(n+1)!} = \frac{1}{n!\,(n+1)} < \frac{1}{n!}$

(ii) $0 < \frac{1}{n!} < \frac{1}{n} \to 0$ so $\frac{1}{n!} \to 0$ as $n \to \infty$

Let's write out a few terms to get a feel for how many terms we might need to use in our approximation:

$s = \frac{1}{0!} - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!} - \frac{1}{7!} + \cdot\ \cdot\ \cdot$

$= 1 - 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120} + \frac{1}{720} - \frac{1}{5040} + \cdot\ \cdot\ \cdot$

Notice that $b_7 = \frac{1}{5040} < \frac{1}{5000} = 0.0002$

and $s_6 = 1 - 1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120} + \frac{1}{720} \approx 0.368056$

By the Alternating Series Estimation Theorem we know that

$$|s - s_6| \leq b_7 < 0.0002$$

This error of less than 0.0002 does not affect the third decimal place, so we have $s \approx 0.368$ correct to three decimal places.

Problem 3: Show that the given series is convergent. Then determine the number of terms needed to find the sum so that the error is less than 0.00005.

$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^6}$