AP Calculus AB
Worksheet
Solving First-Order Differential Equations

Separating Variables

A first-order differential equation in x and y is separable if it can be written so that all the terms involving y are on one side and all the terms involving x are on the other.

A differential equation has variables separable if it is of the form

$$\frac{dy}{dx} = \frac{f(x)}{g(y)} \quad \text{or} \quad g(y)\,dy - f(x)\,dx = 0.$$

The general solution is

$$\int g(y)\,dy - \int f(x)\,dx = C \qquad \text{(C an arbitrary constant)}$$

Example 1: Solve the differential equation $\frac{dy}{dx} = -\frac{x}{y}$, given the initial condition y(0) = 2.

Solution: We first rewrite the equation as

y dy = -x dx

Then we integrate, getting

$$\int y\,dy = -\int x\,dx \qquad \Longrightarrow \qquad \frac{1}{2}y^2 = -\frac{1}{2}x^2 + k$$

or $\quad y^2 + x^2 = C \quad$ (C represents the constant of integration from both sides of the equation and is the result of multiplying both sides by 2)

Since y(0) = 2, we get

$2^2 + 0 = C \quad \Longrightarrow$ C = 4, thus the particular solution is

$$y^2 + x^2 = 4$$

There are two special classes of functions associated with different rates of change that show up often on the AP exam. Recognizing the differential equations and going straight to the general solution can save valuable time on the exam. The two cases are described below.

Case I The rate of change of y is directly proportional to the amount of y present at any time t. This leads to the differential equation

$$\frac{dy}{dt} = k\,y$$

The solution to this differential equaiton is

$$y = C\,e^{kt}$$

Example 2: The population of a country is growing at a rate proportional to its population. If the graowth rate per year is 4% of the current population and intitial population is 100,000 what will the population be in 10 years?

Solution: The differential equation associated with this rate of change is

$\frac{dP}{dt} = 0.04\,P$ so the general solution is

$P = P_0\,e^{0.04\,t}$ and since the initial population is 100,000 (i.e. $P_0 = 100,000$) we have $P = 100,000\,e^{0.04\,t}$ which is the particular solution to the differential equation.

So at t = 10 $P(10) = 100,000\,e^{0.04\times 10} \approx 149,182$ so there will be 149,182 people in 10 years.

Case II The rate of change of a quantity y may be proportional, not to the amount present, but to a difference between that amount and a fixed constant. This leads to the differential equation

$\frac{dy}{dt} = k(A - y)$ The solution to this differential equation is $y = A - C\,e^{kt}$

Example 3: According to Newton's law of cooling, a hot object cools at a rate proportional to the difference between its own temperature and that of its environment. If a roast at room temperature 68°F is put into a 20°F freezer, and if, after 2 hours, the temperature of the roast is 40°F: (a) What is its temperature after 5 hours? (b) How long will it take for the temperature of the roast to fall to 21°F?

Solution: The differential equation associated with this rate of change is

$\frac{dR}{dt} = k(20 - R)$ so the general solution is

R = 20 - C e^{kt} and since the roast is 68°F when t = 0 we have

68 = 20 - C $e^{k\times 0}$ $\Longrightarrow$ C = -48 so we have

R = 20 + 48 e^{kt} also we know that when t = 2 the temperature of the roast is 40°F so

40 = 20 + 48 e^{2k} $\Longrightarrow$ k ≈ -0.437734 so we have

R(t) = 20 + 48 $e^{-0.437734\,t}$ which is the aprticular solution to the differential equation.

So, for (a) R(5) = 20 + 48 $e^{-0.437734\times 5}$ ≈ 25.4°F and for (b) 21 = 20 + 48 $e^{-0.437734\,t}$ and solving for t gives us ≈ 9 hours.

Exercises:

1. Let P(t) represent the number of wolves in a population at time t years, when t ≥ 0. The population P(t) is increasing at a rate directly proportional to 800 - P(t), where the constant of proportionality is k.

(a) If P(0) = 500, find P(t) in terms of t and k.

(b) If P(2) = 700, find k.

(c) Find $\lim_{t \to \infty} P(t)$

2. At any time t ≥ 0, in days, the rate of growth of a bacteria population is given by y' = ky, where k is a constant and y is the number of bacteria present. The initial population is 1,000 and the population triples during the first 5 days.

(a) Write an expression for y at any time t ≥ 0.

(b) By what factor will the population have increased in the first 10 days?

(c) At what time t, in days, will the population have increased by a factor of 6?