AP Calculus AB
Worksheet Areas, Volumes, and Arc Lengths

Areas

To find the area between the graph of \( f(x) \) and the x-axis from \( x = a \) to \( x = b \) we first determine if the function crosses the x-axis on the interval. If it does not cross the x-axis and \( f(x) > 0 \) on the interval then the area is given by

\[
\text{Area} = \int_{a}^{b} f(x) \, dx
\]

If the graph of \( f \) does not cross the x-axis and \( f(x) < 0 \) on the interval then the area is given by

\[
\text{Area} = -\int_{a}^{b} f(x) \, dx
\]

If the graph of \( f \) crosses the x-axis we find the area by integrating on the subintervals defined by the x-intercepts and adding the opposite of the integrals for any region that is under the x-axis. Note: When using your calculator you can take a shortcut when finding the area between the graph of a function and the x-axis by integrating the absolute value of \( f(x) \) on the interval in question. This results in the area between the curve and the x-axis because the graph of the absolute value of \( f(x) \) will lie entirely above the x-axis.

To find the area between two curves we find the points of intersection. If they intersect in only two points then one of the functions will dominate the other on that interval. The height of a typical rectangular element will then be the top function minus the bottom function. So if \( f(x) \) and \( g(x) \) intersect at \( x = a \) and \( x = b \) and \( f(x) > g(x) \) on the interval \([a, b]\) then the area is given by

\[
\text{Area between the graphs of } f(x) \text{ and } g(x) = \int_{a}^{b} [f(x) - g(x)] \, dx
\]
Volumes (Solids of Revolution)

A solid of revolution is obtained when a plane region is revolved about a fixed line, called the axis of revolution. To find the volume of a solid of revolution sketch the region with a typical rectangular element. When this element is revolved about the axis it sweeps out either a disk shape or a washer/ring shape (the washer/ring shape occurs when there is space between the region and the axis of revolution. The volume of a disk is $\pi r^2 h$. The radius of the disk is the height of the rectangular element and the height of the disk is a change along one of the axes (if the axis of revolution is the x-axis or a line parallel to the x-axis the height of the disk is $\Delta x$. If the axis of revolution is the y-axis or a line parallel to the y-axis the height of the disk is $\Delta y$). The volume of the solid consisting of disks is then the sum of the volumes of the individual disks as the number of disks increases indefinitely, giving us a definite integral of the form

$$\text{Volume} = \pi \int_{\text{left endpoint}}^{\text{right endpoint}} (\text{radius})^2 \, dx \text{ (or dy)}$$

The volume of the solid consisting of washers/rings is the sum of the volumes of the individual washers/rings as the number of washers/rings increases indefinitely, giving us a definite integral of the form

$$\text{Volume} = \pi \int_{\text{left endpoint}}^{\text{right endpoint}} [(\text{outside radius})^2 - (\text{inside radius})^2] \, dx \text{ (or dy)}$$

Volumes (With Known Cross Sections)

If the area of a cross section of a solid is known and can be expressed in terms of $x$, then the volume of a typical slice can be determined. The volume of the solid is obtained, as usual, by letting the number of slices increase indefinitely (As you have probably already surmised finding the volume of a solid of revolution is really a special case of a volume of a solid of known cross sections). The volume of a solid with known cross sections is the sum of the individual slices as the number of slices increases indefinitely and can be expressed as the following definite integral

$$\text{Volume} = \int_{\text{left endpoint}}^{\text{right endpoint}} (\text{cross sectional area}) \, dx \text{ (or dy)}$$
Arc Length

If the derivative of a function \( y = f(x) \) is continuous on the interval \([a, b]\), then the length \( s \) of the arc of the curve of \( y = f(x) \) from the point where \( x = a \) to the point where \( x = b \) is given by

\[
s = \int_a^b \sqrt{1 + \left( \frac{dy}{dx} \right)^2} \, dx \quad \text{or} \quad \int_a^b \sqrt{1 + (f'(x))^2} \, dx
\]

Although the formula for the arc length of a curve is easy to apply, the integrands it generates often do not have elementary antiderivatives. Arc length problems typical show up on the calculator active portion of the AP exam.

Exercises

1. Find the area enclosed by the graphs of \( y = x^2 - 3 \) and \( y = 1 \).

2. Find the area of the region bounded by the curve \( y = \sin x \) and \( y = \cos x \) from \( x = \pi/4 \) to \( 5\pi/4 \).

3. Find the volume of the solid generated when the region bounded by \( y = x^2 \), \( x = 2 \), and \( y = 0 \) is revolved about the x-axis.
4. Find the volume of the solid generated when the region bounded by \( y = x^2 \), \( x = 2 \), and \( y = 0 \) is revolved about the y-axis.

5. Find the volume of the solid generated when the region bounded by \( y = 3x - x^2 \) and \( y = x \) is revolved about the x-axis.

6. Find the volume of the solid generated when the region bounded by \( y = 3x - x^2 \) and \( y = x \) is revolved about the line \( y = -2 \).

7. The base of a solid is the region bounded by \( y = e^{-x} \), the x-axis, the y-axis, and the line \( x = 1 \). Each cross section perpendicular to the x-axis is a square. What is the volume of the solid?

8. (Calculator active) Set up an evaluate an integral to compute the arc length of the curve \( y = x^3 \) from \( x = 0 \) to \( x = 5 \).